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$su(1, 1)$-Barut–Girardello coherent states for Landau levels

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Abstract
It is shown that the Hilbert space corresponding to all the quantum states of the Landau problem can be split in two different ways: as infinite direct sums of the finite- and infinite-dimensional representation subspaces of the Lie algebras $su(2)$ and $su(1, 1)$ with finite- and infinite-fold degeneracies, respectively. For each of the Hilbert representation subspaces of the Lie algebra $su(1, 1)$, we construct a suitable linear combination of its bases as the Barut–Girardello coherent states.

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1. Introduction

The physics of charged particles in a magnetic field has been one of the important problems in quantum mechanics, inspired by condensed matter physics, quantum optics etc. On the other hand, the variants, generalizations and applications of coherent states have been extensively studied over the last four decades. A comprehensive review of this development can be found in [1–4]. Consideration of coherent states for a charged particle is one of the interesting problems in various fields of physics [5]. Coherent states were considered recently for the system of a two-dimensional fermion gas in a constant magnetic field [6]. In [7] we constructed the generalized Kluder–Perelomov [1, 2] and Gazeau–Kluder [8–10] coherent states of Landau levels using two different representations for the Lie algebra h_4. It should be recalled that the Landau problem [11] is related to the motion of a charged particle on the flat plane xy in the presence of a constant magnetic field along the z-axis. Here we reorganize the Landau levels into two different hidden symmetries, namely $su(2)$ and $su(1, 1)$. The representation of $su(1,1)$ by the Landau levels then leads to the construction of the Barut–Girardello [12] coherent states.

In order to provide the necessary mathematical tools, in this section we explain some results related to the splitting of all quantum states of the Landau problem [11] in two different ways into infinite direct sums of the representation spaces of the Heisenberg Lie algebra h_4. In the previous work [7], by introducing the associated Laguerre functions as

$$L_{n,m}^{(\alpha,\beta)}(x) = \frac{(-1)^m\sqrt{\frac{\beta^{\alpha+m+1}}{\Gamma(n-m+1)\Gamma(n+\alpha+1)}}}{x^{\alpha+\frac{m}{2}}\,\mathrm{e}^{-\beta x}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m}(x^{n+\alpha}\,\mathrm{e}^{-\beta x}) \tag{1}$$

with $\beta > 0, \alpha > -1, n \geqslant 0, 0 \leqslant m \leqslant n$, we showed that the set of three-dimensional harmonic oscillator quantum states

$$\psi_{n,m}(r) = \left(\frac{r}{2}\right)^{\frac{2\alpha+1}{2}}\mathrm{e}^{-\frac{\beta}{8}r^2}L_{n,m}^{(\alpha,\beta)}\left(\frac{r^2}{4}\right) \tag{2}$$

form an orthonormal set with the same m but with different ns:

$$\int_0^\infty \psi_{n,m}(r)\psi_{n',m}(r)\,\mathrm{d}r = \delta_{nn'}. \tag{3}$$

Also, we constructed the Hilbert space $\mathcal{H} := \mathrm{span}\,\{|n,m\rangle\}_{n\geqslant 0, 0\leqslant m\leqslant n}$ from the bases

$$|n,m\rangle = \frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}\psi_{n,m}(r) \tag{4}$$

with $0 \leqslant \varphi < 2\pi$, as quantum states of the Landau problem corresponding to the motion of a spinless charged particle on a flat surface in the presence of a constant magnetic field $\beta/2$ along the z-axis. Using equation (3), it becomes obvious that the bases of the Hilbert space $\mathcal{H}$ with respect to the following inner product constitute an orthonormal set for different ns and ms:

$$\langle n,m|n',m'\rangle = \int_{\varphi=0}^{2\pi}\int_{r=0}^{\infty}\left(\frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}\psi_{n,m}(r)\right)^*\left(\frac{\mathrm{e}^{\mathrm{i}m'\varphi}}{\sqrt{2\pi}}\psi_{n',m'}(r)\right)\mathrm{d}r\,\mathrm{d}\varphi = \delta_{nn'}\delta_{mm'}. \tag{5}$$

The orthonormality relation (5) imposes the completeness relation in the Hilbert space $\mathcal{H}$ as

$$\sum_{n=0}^{+\infty}\sum_{m=0}^{n}|n,m\rangle\langle n,m| = I_{\mathcal{H}}. \tag{6}$$

In figure 1 of [7], we schematically showed the bases of the Hilbert space $\mathcal{H}$ as the points (n,m), with $0 \leqslant m \leqslant n$, on a flat plane whose horizontal and vertical axes are labelled with n and m, respectively.

There we also extracted two classes of generators of the Heisenberg Lie algebra h_4 as

$$\begin{aligned}
L_+ &= \mathrm{e}^{\mathrm{i}\varphi}\left(\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial\varphi} + \frac{\beta}{4}r - \frac{2\alpha+1}{2r}\right)\\
L_- &= \mathrm{e}^{-\mathrm{i}\varphi}\left(-\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial\varphi} + \frac{\beta}{4}r - \frac{2\alpha-1}{2r}\right)\\
L_3 &= -\mathrm{i}\frac{\partial}{\partial\varphi} \qquad I = 1
\end{aligned} \tag{7}$$

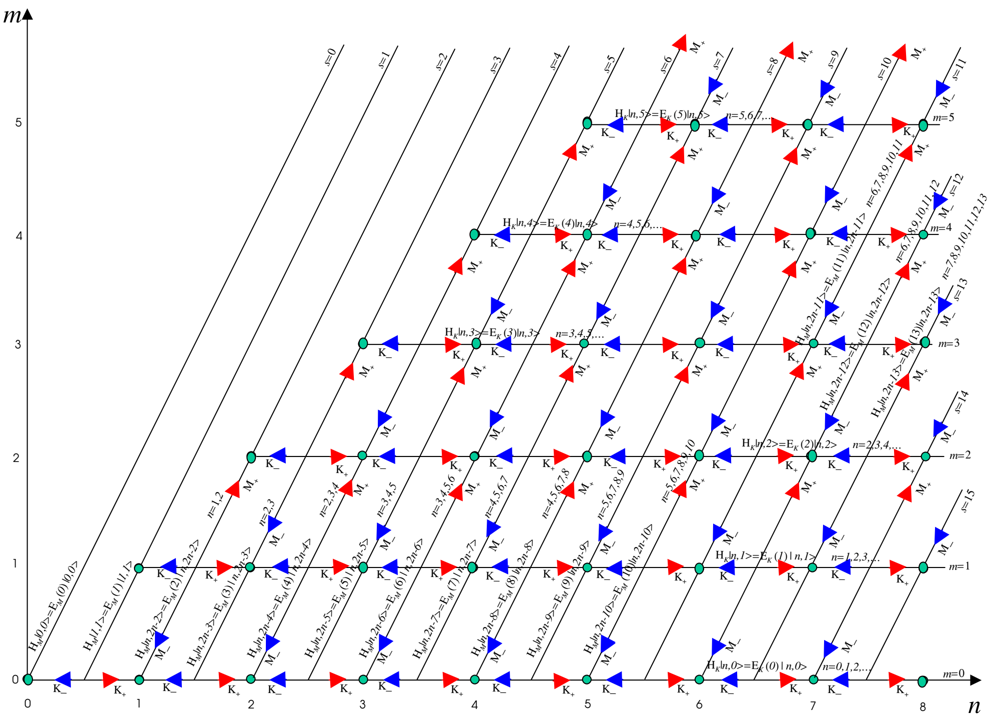

Figure 1. The Landau quantum states lattice (n, m) as the bases of the Hilbert representation space $\mathcal{H}$.

and

$$J_+ = e^{i\varphi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi} - \frac{\beta}{4}r - \frac{2\alpha+1}{2r}\right)$$
$$J_- = e^{-i\varphi}\left(-\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi} - \frac{\beta}{4}r - \frac{2\alpha-1}{2r}\right) \tag{8}$$
$$J_3 = L_3 = -i\frac{\partial}{\partial\varphi} \qquad I = 1$$

satisfying the commutation relations

$$[L_+, L_-] = \beta I \qquad [L_3, L_\pm] = \pm L_\pm \qquad [\mathbf{L}, I] = 0 \tag{9}$$

and

$$[J_+, J_-] = -\beta I \qquad [J_3, J_\pm] = \pm J_\pm \qquad [\mathbf{J}, I] = 0 \tag{10}$$

respectively. The bases of the Hilbert space $\mathcal{H}$ represent the generators $\{L_+, L_-, L_3, I\}$ and $\{J_+, J_-, J_3, I\}$ in two different ways, namely,

$$L_+|n, m-1\rangle = \sqrt{(n-m+1)\beta}|n, m\rangle$$
$$L_-|n, m\rangle = \sqrt{(n-m+1)\beta}|n, m-1\rangle \tag{11}$$
$$L_3|n, m\rangle = m|n, m\rangle \qquad I|n, m\rangle = |n, m\rangle$$

and

$$J_+|n-1,m-1\rangle = \sqrt{(n+\alpha)\beta}|n,m\rangle$$
$$J_-|n,m\rangle = \sqrt{(n+\alpha)\beta}|n-1,m-1\rangle \qquad (12)$$
$$J_3|n,m\rangle = m|n,m\rangle \qquad I|n,m\rangle = |n,m\rangle$$

as finite- and infinite-dimensional representations, respectively. There, labelling the oblique lines with $d = 1, 2, 3, \ldots,$ we wrote the equation of the dth line as $n = m + d - 1$. Then, defining two classes of Hilbert subspaces, $\mathcal{H}_n := \text{span}\{|n,m\rangle\}_{0\leqslant m\leqslant n}$ and $\mathcal{H}_d := \text{span}\{|n, n-d+1\rangle\}_{n\geqslant d-1}$, we concluded that $\left\{\mathcal{H}_n \bigcap \mathcal{H}_{n'} = 0 \text{ for } n \neq n', \mathcal{H} = \oplus_{n=0}^{\infty}\mathcal{H}_n\right\}$ and $\left\{\mathcal{H}_d \bigcap \mathcal{H}_{d'} = 0 \text{ for } d \neq d', \mathcal{H} = \oplus_{d=1}^{\infty}\mathcal{H}_d\right\}$. Moreover, upon referring to figure 1 of [7], one can see that the number of quantum states lying on the nth vertical line is $n + 1$, while on the dth oblique line there are an infinite number of them. In other words: $\dim\mathcal{H}_n = n + 1$ and $\dim\mathcal{H}_d = \infty$, i.e. $\mathcal{H}_n$ and $\mathcal{H}_d$ constitute the finite- and infinite-dimensional representation spaces of the Heisenberg Lie algebra h_4 corresponding to the generators $\{L_+, L_-, L_3, I\}$ and $\{J_+, J_-, J_3, I\}$, respectively. Also, using the explicit forms of the generators $L_\pm$ and $J_\pm$, we conclude that the diagram given in figure 1 of [7] is commutative; that is,

$$[J_+, L_\pm] = [J_-, L_\pm] = 0. \qquad (13)$$

Furthermore, from equations (11) and (12), we conclude that $L_\pm : \mathcal{H}_d \to \mathcal{H}_{d\mp1}$ and $J_\pm : \mathcal{H}_n \to \mathcal{H}_{n\pm1}$. Note that the inner product (5) defined by the inner products of the bases in the Hilbert space $\mathcal{H}$ induces two inner products in the Hilbert subspaces $\mathcal{H}_n$ and $\mathcal{H}_d$ as $\langle n,m|n,m'\rangle = \delta_{mm'}$ with $0 \leqslant m,m' \leqslant n$ and $\langle n, n-d+1|n', n'-d+1\rangle = \delta_{nn'}$ with $n, n' \geqslant d-1$, respectively. These orthonormality relations immediately give the completeness relations in the Hilbert subspaces $\mathcal{H}_n$ and $\mathcal{H}_d$ as $\sum_{m=0}^{n}|n,m\rangle\langle n,m| = I_{\mathcal{H}_n}$ and $\sum_{n=d-1}^{\infty}|n,n-d+1\rangle\langle n,n-d+1| = I_{\mathcal{H}_d}$. The Casimir operators corresponding to the set of generators $\{L_+, L_-, L_3, I\}$ and $\{J_+, J_-, J_3, I\}$, i.e.

$$H_L = \frac{1}{2}\left[L_+L_- + \beta L_3 - \frac{\beta}{2}\right] \qquad (14)$$

and

$$H_J = \frac{1}{2}\left[J_+J_- - \beta J_3 + \frac{\beta}{2}\right] \qquad (15)$$

satisfy the following eigenvalue equations:

$$H_L|n,m\rangle = \frac{\beta}{2}\left(n + \frac{1}{2}\right)|n,m\rangle \qquad 0 \leqslant m \leqslant n \qquad (16)$$

and

$$H_J|n, n-d+1\rangle = \frac{\beta}{2}\left(d + \alpha - \frac{1}{2}\right)|n, n-d+1\rangle \qquad n \geqslant d-1 \qquad (17)$$

on the Hilbert subspaces $\mathcal{H}_n$ and $\mathcal{H}_d$, respectively. As mentioned in [7], the Casimir operators H_L and H_J contain the constant magnetic field $\beta/2$ in the negative and positive directions of the z-axis with $(n + 1)$-fold and infinite-fold degeneracies, respectively.

2. Splitting of the Landau levels as representation subspaces of the Lie algebras $su(2)$ and $su(1,1)$

Let us define the second- and first-order differential operators

$$M_+ := \frac{1}{\beta}J_+L_+ \qquad M_- := \frac{1}{\beta}J_-L_- \qquad M_3 := \frac{1}{2\beta}(J_+J_- - L_-L_+). \qquad (18)$$

Using the commutation relations of the Lie algebra h_4 as (9) and (10) and also the commutation relations (13), one can easily derive the commutation relations of the Lie algebra $su(2)$ [13] for the generators M_+, M_- and M_3:

$$[M_+, M_-] = 2M_3 \qquad [M_3, M_\pm] = \pm M_\pm. \tag{19}$$

The following representation for the Lie algebra $su(2)$ in the Hilbert space $\mathcal{H}$ can be immediately found using the representations (11) and (12) of the Lie algebra h_4:

$$\begin{aligned} M_+|n-1, m-2\rangle &= \sqrt{(n+\alpha)(n-m+1)}|n, m\rangle \\ M_-|n, m\rangle &= \sqrt{(n+\alpha)(n-m+1)}|n-1, m-2\rangle \\ M_3|n, m\rangle &= \tfrac{1}{2}(m+\alpha)|n, m\rangle. \end{aligned} \tag{20}$$

Figure 1 of this paper shows decompositions of the Hilbert space $\mathcal{H}$ into distinct classes of the Hilbert subspaces made in two different ways, which are also different from those in figure 1 of [7]. The oblique and the horizontal lines indicate these two types of decomposition of the Hilbert space schematically. If we label the oblique lines with s, where $s = 0, 1, 2, \ldots,$ then we can write their equations as $m = 2n - s$. Then we define the new finite-dimensional Hilbert subspaces as $\mathcal{H}_s := \mathrm{span}\{|n, 2n-s\rangle\}_{s+1-\dim \mathcal{H}_s \leqslant n \leqslant s}$. In fact, the Hilbert subspace $\mathcal{H}_s$ involves all the Landau levels lying on the sth oblique line. Note that the limitation $0 \leqslant m \leqslant n$ on the Hilbert subspace $\mathcal{H}_s$ leads to the limitation $\frac{s}{2} \leqslant n \leqslant s$. For s even and odd, i.e. $s = 2k$ and $s = 2k+1$, we deduce that $\dim \mathcal{H}_{2k} = s - k + 1 = \left[\frac{s}{2} + 1\right]$ and $\dim \mathcal{H}_{2k+1} = s - k = \left[\frac{s}{2} + 1\right]$, respectively. Here the symbol [ ] means the integer part. Therefore, for every arbitrary s we have $\dim \mathcal{H}_s = \left[\frac{s}{2} + 1\right]$. Now, it is clear that $\left\{\mathcal{H}_s \bigcap \mathcal{H}_{s'} = 0 \text{ for } s \neq s', \mathcal{H} = \oplus_{s=0}^{\infty} \mathcal{H}_s\right\}$. The orthonormality and completeness relations in the Hilbert subspaces $\mathcal{H}_s$ are inherited from the corresponding relations in the Hilbert space $\mathcal{H}$. That is, equations (5) and (6) give

$$\langle n, 2n-s|n', 2n'-s\rangle = \delta_{nn'} \qquad n, n' = s, s-1, \ldots, s+1-\left[\frac{s}{2}+1\right] \tag{21}$$

and

$$\sum_{n=s+1-[\frac{s}{2}+1]}^{s} |n, 2n-s\rangle\langle n, 2n-s| = I_{\mathcal{H}_s}. \tag{22}$$

One may obtain a representation of the Lie algebra $su(2)$ in the Hilbert subspace $\mathcal{H}_s$ using the relations (20) as follows:

$$\begin{aligned} M_+|n-1, 2n-s-2\rangle &= \sqrt{(n+\alpha)(n+s+1)}|n, 2n-s\rangle \\ M_-|n, 2n-s\rangle &= \sqrt{(n+\alpha)(n+s+1)}|n-1, 2n-s-2\rangle \\ M_3|n, 2n-s\rangle &= \tfrac{1}{2}(2n-s+\alpha)|n, 2n-s\rangle. \end{aligned} \tag{23}$$

The Casimir operator of the Lie algebra $su(2)$,

$$H_M := M_+M_- + M_3^2 - M_3 \tag{24}$$

satisfies the following eigenvalue equations on the Hilbert subspaces $\mathcal{H}_s$:

$$H_M|n, 2n-s\rangle = E_M(s)|n, 2n-s\rangle \qquad n = s, s-1, \ldots, s+1-\left[\frac{s}{2}+1\right] \tag{25}$$

where

$$E_M(s) = \frac{s+\alpha}{2}\left(\frac{s+\alpha}{2}+1\right). \tag{26}$$

Therefore, the Hilbert subspace $\mathcal{H}_s$ as a representation space for the Lie algebra $su(2)$ contains an $\left[\frac{s}{2}+1\right]$-fold degeneracy. This means that all the Landau quantum states lying on the sth oblique line have the same spectrum for the Casimir operator of the Lie algebra $su(2)$. We can also consider the state $|s, s\rangle$ as the highest state, which is annihilated by the raising operator M_+:

$$M_+|s, s\rangle = 0. \tag{27}$$

The solution of the differential equation (27) can be computed in the following form, in agreement with the analytic solution (4):

$$|s, s\rangle = \frac{(-1)^s}{\sqrt{2\pi}}\sqrt{\frac{\beta^{\alpha+s+1}}{\Gamma(\alpha+s+1)}}\,\mathrm{e}^{\mathrm{i}s\varphi}\left(\frac{r}{2}\right)^{\frac{2\alpha+2s+1}{2}}\,\mathrm{e}^{-\frac{\beta}{8}r^2}. \tag{28}$$

With the help of the second equation of (23), an arbitrary state of the Hilbert subspace $\mathcal{H}_s$ (the sth oblique line) can be calculated in an algebraic manner as follows:

$$|n, 2n-s\rangle = \sqrt{\frac{\Gamma(\alpha+n+1)\Gamma(n+s+2)}{\Gamma(\alpha+s+1)\Gamma(2s+2)}}M_-^{s-n}|s, s\rangle$$
$$n = s, s-1, \ldots, s+1-\left[\frac{s}{2}+1\right]. \tag{29}$$

Therefore, the Hilbert space $\mathcal{H}$ can be decomposed into the infinite direct sum of the finite-dimensional Hilbert subspaces $\mathcal{H}_s$, which represent the Lie algebra $su(2)$ with $\left[\frac{s}{2}+1\right]$-fold degeneracy.

Let us now define the new second-order differential operators

$$K_+ := \frac{1}{\beta}J_+L_- \qquad K_- := \frac{1}{\beta}J_-L_+ \qquad K_3 := \frac{1}{2\beta}(J_-J_+ + L_-L_+). \tag{30}$$

With the help of the commutation relations (9), (10) and (13), it is found that the generators K_+, K_- and K_3 satisfy the commutation relations of the Lie algebra $su(1, 1)$ [13] as follows:

$$[K_+, K_-] = -2K_3 \qquad [K_3, K_\pm] = \pm K_\pm. \tag{31}$$

According to equations (11) and (12), the representation of the Lie algebra $su(1, 1)$ in the Hilbert space $\mathcal{H}$ is found to be of the following form:

$$\begin{aligned} K_+|n-1, m\rangle &= \sqrt{(n+\alpha)(n-m)}|n, m\rangle \\ K_-|n, m\rangle &= \sqrt{(n+\alpha)(n-m)}|n-1, m\rangle \\ K_3|n, m\rangle &= \tfrac{1}{2}(2n-m+\alpha+1)|n, m\rangle. \end{aligned} \tag{32}$$

We now construct a new class of the Hilbert subspaces which not only realize a representation of the Lie algebra $su(1, 1)$ but also split the Landau levels into an infinite direct sum of subspaces, in which the binary intersections are empty. The horizontal lines are specified by the values of m, where $m = 0, 1, 2, \ldots$. This leads us to introduce the infinite-dimensional Hilbert subspaces $\mathcal{H}_m := \mathrm{span}\{|n, m\rangle\}_{n\geqslant m}$. At this stage one can easily see that $\left\{\mathcal{H}_m \bigcap \mathcal{H}_{m'} = 0\right.$ for $m \neq m'$, $\left.\mathcal{H} = \oplus_{m=0}^{\infty}\mathcal{H}_m\right\}$. It must therefore be noted that the relations (32) describe the representation of the Lie algebra $su(1, 1)$ in the Hilbert subspaces $\mathcal{H}_m$. It has to be emphasized that according to equations (20) and (32), the ladder operators of the Lie algebras $su(2)$ and $su(1, 1)$ shift the infinite- and finite-dimensional Hilbert subspaces as $M_\pm : \mathcal{H}_m \to \mathcal{H}_{m\pm 2}$ and $K_\pm : \mathcal{H}_s \to \mathcal{H}_{s\pm 2}$. Again the orthonormality and the completeness relations in the Hilbert subspaces $\mathcal{H}_m$ are deduced from (5) and (6) to be

$$\langle n, m|n', m\rangle = \delta_{nn'} \qquad n, n' = m, m+1, m+2, \ldots \tag{33}$$

and

$$\sum_{n=m}^{+\infty}|n, m\rangle\langle n, m| = I_{\mathcal{H}_m}. \tag{34}$$

The eigenvalue equations for the Casimir operator of the Lie algebra $su(1,1)$,

$$H_K := K_+K_- - K_3^2 + K_3 \tag{35}$$

using the representation equations (32), are calculated as follows:

$$H_K|n,m\rangle = E_K(m)|n,m\rangle \qquad n = m, m+1, m+2, \ldots \tag{36}$$

where

$$E_K(m) = -\frac{m+\alpha+1}{2}\left(\frac{m+\alpha+1}{2} - 1\right). \tag{37}$$

Thus, all the quantum states on a horizontal line have the same spectrum for the Casimir operator of the Lie algebra $su(1,1)$ which means that the degeneracy of the representation spaces $\mathcal{H}_m$ is infinite-fold. The lowest state in the Hilbert subspace $\mathcal{H}_m$ is $|m,m\rangle$, which is annihilated by the lowering operator K_-:

$$K_-|m,m\rangle = 0. \tag{38}$$

The solution of (38) is just (28), provided that s is replaced by m. The arbitrary states belonging to the Hilbert subspace $\mathcal{H}_m$ (the mth horizontal line) are calculated in the following form by means of the first equation of (32):

$$|n,m\rangle = \sqrt{\frac{\Gamma(\alpha+m+1)}{\Gamma(n-m+1)\Gamma(n+\alpha+1)}}K_+^{n-m}|m,m\rangle \qquad n = m, m+1, m+2, \ldots. \tag{39}$$

Consequently, the Hilbert space $\mathcal{H}$ splits into an infinite direct sum of the infinite-dimensional Hilbert subspaces $\mathcal{H}_m$ which represent the Lie algebra $su(1,1)$ with infinite-fold degeneracy.

3. $su(1,1)$-Barut–Girardello coherent states in the Hilbert subspaces $\mathcal{H}_m$ of the Landau levels

Let us introduce the Barut–Girardello coherent states [12] in the Hilbert subspaces $\mathcal{H}_m$ as eigenstates of the lowering generator K_- of the Lie algebra $su(1,1)$:

$$K_-|z\rangle_m = z|z\rangle_m \tag{40}$$

in which z is an arbitrary complex variable (with the polar form $z = r\,\mathrm{e}^{\mathrm{i}\phi}$, $0 \leqslant r < \infty$, $0 \leqslant \phi < 2\pi$). Using the second equation of (32), we can calculate the coherent states $|z\rangle_m$ as the linear combinations of all the orthonormal basis states of the Hilbert subspace $\mathcal{H}_m$:

$$|z\rangle_m = \frac{|z|^{\frac{\alpha+m}{2}}}{\sqrt{I_{\alpha+m}(2|z|)}}\sum_{n=m}^{+\infty}\frac{z^{n-m}|n,m\rangle}{\sqrt{\Gamma(n-m+1)\Gamma(\alpha+n+1)}} \tag{41}$$

where $I_{\alpha+m}(2|z|)$ is the modified Bessel function of the first kind as follows [14]:

$$I_{\alpha+m}(2|z|) = \sum_{n=0}^{+\infty}\frac{|z|^{\alpha+2n+m}}{\Gamma(n+1)\Gamma(\alpha+n+m+1)}. \tag{42}$$

Meanwhile, by means of the above-described degeneracy in equation (36), it is easy to see that the coherent states of $\mathcal{H}_m$, i.e. $|z\rangle_m$, given in (41) satisfy the following eigenvalue equations:

$$H_K|z\rangle_m = E_K(m)|z\rangle_m. \tag{43}$$

In order to complete the discussion as in references [1, 12], we introduce an appropriate measure $\mathrm{d}\sigma(z)$ such that the property of resolution of the identity is realized for the coherent states $|z\rangle_m$ in the Hilbert space $\mathcal{H}_m$:

$$\int \mathrm{d}\sigma(z)|z\rangle_{m\,m}\langle z| = I_{\mathcal{H}_m}. \tag{44}$$

Some calculation shows that on choosing

$$\mathrm{d}\sigma(z) = \frac{2}{\pi} I_{\alpha+m}(2r) K_{\frac{\alpha+m}{2}}(2r) r \,\mathrm{d}r \,\mathrm{d}\phi \tag{45}$$

using the completeness relation (34) of $\mathcal{H}_m$, the resolution of the identity condition (44) is realized. The function $K_{\frac{\alpha+m}{2}}(2r)$ is the modified Bessel function of the second kind [14]:

$$K_{\frac{\alpha+m}{2}}(2r) = \frac{\pi}{2\sin\left(\frac{(\alpha+m)\pi}{2}\right)} \left(I_{-\frac{\alpha+m}{2}}(2r) - I_{\frac{\alpha+m}{2}}(2r)\right) \tag{46}$$

which satisfies the following integral relation:

$$4\int_0^\infty r^{2n+\alpha-m+1} K_{\frac{\alpha+m}{2}}(2r)\,\mathrm{d}r = \Gamma(n-m+1)\Gamma(n+\alpha+1). \tag{47}$$

Also, the inner product of the coherent states $|z\rangle_m$ and $|z'\rangle_{m'}$ belonging to the Hilbert subspaces $\mathcal{H}_m$ and $\mathcal{H}_{m'}$ is calculated as

$${}_{m'}\langle z'|z\rangle_m = \delta_{m'm}\left(\frac{|z'z|}{\bar{z}'z}\right)^{\frac{\alpha+m}{2}} \frac{I_{\alpha+m}(2\sqrt{\bar{z}'z})}{\sqrt{I_{\alpha+m}(2|z'|)I_{\alpha+m}(2|z|)}}. \tag{48}$$

Relation (48) implies that not only are the coherent states of the different representation subspaces of the Lie algebra $su(1,1)$ orthogonal, but also they are normalized to unity with respect to the inner product (33) in the Hilbert subspace $\mathcal{H}_m$: ${}_m\langle z|\, z\rangle_m = 1$. Furthermore, one can readily conclude that, according to the Mandel criterion, the weight distribution functions of the coherent states $|z\rangle_m$ are of the Poissonian and sub-Poissonian types.

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